HENSELIAN LOCAL RINGS

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1. April 6th Henselian local rings

1.1 Notation
A triple \((A, m, k)\) means a commutative local ring \(A\) with maximal ideal \(m\) and residue field \(k = A/m\). Overlines mean “reduction modulo \(m\”).

1.2 Lemma
Hensel. Let \(p\) be a prime. The ring of \(p\)-adic integers is \((\mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{F}_p)\). Suppose

1. \(f \in \mathbb{Z}_p[X], \overline{f} \in \mathbb{F}_p[X], f\) is monic.
2. \(a_0 \in \mathbb{F}_p, \overline{f}(a_0) = 0\) and \(\overline{f'}(a_0) \neq 0\).

Then there exists an \(a \in \mathbb{Z}_p\) such that \(f(a) = 0\) and \(\overline{a} = a_0\).
This is a special case of Theorem 1.7.

1.3 Example
The polynomial \(f(X) = X^2 - 2\) has a solution in \(\mathbb{Z}_7\).

Proof. Since \(\overline{f}(X) = X^2 - 2\) has a solution 3 in its residue field \(\mathbb{F}_7\) and \(\overline{f}(X) = 2X\) does not have solution 3. By Hensel’s lemma, the equation has a solution \(3 + a_17 + a_27^2 + \cdots\) in \(\mathbb{Z}_7\).

Not only the existence, Hensel’s lemma can provide the solution. Next we find \(a_1\).

\[
0 = (3 + a_17 + a_27^2 + \cdots)^2 - 2 = (1 + 6a_1)7 + (a_1^2 + 6a_2)7^2 + \cdots \in \mathbb{Z}_7,
\]

it suffices to solve \(1 + 6a_1 = 0\) in \(\mathbb{F}_7\), we can take \(a_1 = 1\). Then

\[
0 = (3 + a_17 + a_27^2 + \cdots)^2 - 2 = (2 + 6a_2)7^2 + \cdots \in \mathbb{Z}_7,
\]

it suffices to solve \(2 + 6a_2 = 0\) in \(\mathbb{F}_7\), we can take \(a_2 = 2, \ldots\).

Therefore the solution is \(3 + 1 \cdot 7 + 2 \cdot 7^2 + \cdots \in \mathbb{Q}_7\) \(\square\)

The following are from [Sta17, Tag 03QD]

1.4 Definition
Let \((A, m, k)\) be a local ring. We call it henselian if we suppose

1. \(f \in A[X], \overline{f} \in k[X], f\) is monic.
2. \(a_0 \in k, \overline{f}(a_0) = 0\) and \(\overline{f'}(a_0) \neq 0\).

Then there exists a unique \(a \in A\) such that \(f(a) = 0\) and \(\overline{a} = a_0\).

1.5 Example
Every field is a henselian local ring since its maximal ideal is 0 and its residue field is itself.

1.6 Example
\(\mathbb{Z}_p\) and \(k[[T]]\) are henselian, where \(k\) is a field.

1.7 Theorem
A complete Hausdorff local ring \((A, m, k)\) is henselian.

Proof. Newton. Suppose \(f(T) \in A[T]\) is monic, \(a_0 \in A/m\) such that \(\overline{f}(a_0) = 0\) and \(\overline{f'}(a_0) \neq 0\). We will construct inductively \(x_n \in A\) such that

1. \(\overline{x_n} = a_0; (2) f(x_n) \in m^{n+1}; (3) f'(x_n) \in A^\star\).

(1a) Since the quotient map \(A \to k\) is surjective, there exists \(x_0 \in A\) such that \(\overline{x_0} = a_0\).

(2a) Since \(\overline{f}(x_0) = \overline{f}(a_0) = 0\), \(f(x_0) \in m\).

(3a) Since \(\overline{f'}(a_0) \neq 0\), \(f'(x_0) \in A \setminus m = A^\star\).

Suppose \(x_n\) is constructed. Let \(x_{n+1} = x_n - f(x_n)/f'(x_n)\).
\[1 \text{ Proposition} \]

\[\frac{1}{n+1} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = a_0 \quad \text{by (1 \_n)(2 \_n)}. \]

\[2 \text{ Proposition} \]

The Taylor expansion of \( f(x) \) at \( x_n \) is

\[
f(x) = \sum_{i=0}^{d} \frac{f^{(i)}(x_n)}{i!} (x - x_n)^i, \quad d = \deg(f).
\]

Therefore \( f(x_{n+1}) = \sum_{i=0}^{d} \frac{f^{(i)}(x_n)}{i!} (x_{n+1} - x_n)^i \). Since \( x_{n+1} - x_n = -f(x_n)/f'(x_n) \),

\[
f(x_{n+1}) = \sum_{i=2}^{d} \frac{f^{(i)}(x_n)}{i!} (x_{n+1} - x_n)^i. \quad \text{By (2\_n), } x_{n+1} - x_n = -f(x_n)/f'(x_n) \in m^{n+1}.
\]

Therefore \( f(x_{n+1}) \in m^{2(n+1)} \subset m^{n+2} \) for all \( n \geq 0 \).

\[3 \text{ Proposition} \]

Since \( x_{n+1} - x_n \in m \) and it is a factor of \( f'(x_{n+1}) - f'(x_n) \), we have \( f'(x_{n+1}) - f'(x_n) \in m \). By (3\_n) \( f'(x_n) \in A \setminus m \). Hence \( f'(x_{n+1}) \in A \setminus m = A^* \).

Now, we return to the proof of the theorem. The m-adic topology is Hausdorff means that \( \bigcap_{n=0}^{\infty} m^{n+1} = 0 \). By (2\_n), \( (x_n) \) is a Cauchy sequence. Since \( A \) is m-adically complete, there exists \( \lim_{n \to \infty} x_n = a \in A \). By (1\_n), \( \bar{a} = \bar{a} \).

Finally,

\[
f(a) = \lim_{n \to \infty} f(x_n) \quad \text{Polynomial } f \text{ is continuous}
\]

\[
\in \bigcap_{n=0}^{\infty} m^{n+1} \quad \text{By (2\_n)}
\]

\[
= 0 \quad \text{Since } A \text{ is Hausdorff}.
\]

\[\Box\]

**1.8 Lemma**

Let \( A \) be a ring with prime ideal \( p \). For all ring \( B, (A \times B)_{p \times B} \simeq A_p \).

**Proof.** First, \( p \times B \) is a prime ideal of \( A \times B \) since \( \frac{A \times B}{p \times B} \simeq \frac{A}{p} \) is an integral domain. Let \( f: (A \times B)_{p \times B} \to A_p, f \left( \frac{(a,b)}{(s,b')} \right) = \frac{a}{s} \).

- It is well-defined. Suppose \( \frac{(a,b)}{(s,b')} = \frac{(a',c)}{(s',c')} \). Then there exists \( (s'',d) \in (A \setminus p) \times B \) such that \( (s'',d)((s',c')(a,b) - (s,b')(a',c)) = (0,0) \). Then \( s''(sa - s'a) = 0 \) and hence \( \frac{a}{s} = \frac{a'}{s'} \).

- It is surjective by definition.

- It is injective. Suppose \( \frac{(a,b)}{(s,b')} = f \left( \frac{(a',c)}{(s',c')} \right) \). Then \( \frac{a}{s} = \frac{a'}{s'} \). Then there exists \( s'' \in A \setminus p \) such that \( s''(sa - s'a) = 0 \). Hence \( (s',0)((s',c')(a,b) - (s,b')(a',c)) = (s''(sa - s'a),0(c'b - b'c)) = (0,0) \).

i.e. \( \frac{(a,b)}{(s,b')} = \frac{(a',c)}{(s',c')} \) in \( (A \times B)_{p \times B} \).

\[\Box\]

**1.9 Proposition**

Let \( (A,\mathfrak{m},k) \) be a local ring. Let \( B \) be a finite \( A \)-algebra. (By [Bou89, Ch. V, § 2, Pro. 1, Prop. 3], it is a semi-local ring whose maximal ideals \( n_1, \ldots, n_k \) lie over
1.10 Proposition

Let \((A, m, k)\) be a local ring. Let \(B\) be a finite \(A\)-algebra. Then

(1) \(\text{Idem}(B) \rightarrow \text{Idem}(\overline{B})\) is injective, where \(\overline{B} = B \otimes_A k\).

(2) \(\text{Idem}(B) \rightarrow \text{Idem}(\overline{B})\) is bijective iff \(B\) is a product of local rings.

Proof. (1) Suppose \(e, e' \in \text{Idem}(B)\) such that \(e = e'\). Let \(x = e - e'\).

\[
x^3 = e^3 - 3ee' + 3e'e^2 - e'^3 = e - 3ee' + 3e'e - e' = e - e' = x.
\]

Then \(x(1 - x^2) = 0\). Since \(\overline{x} = 0\), \(x \in mB \subset \bigcap_{i=1}^k n_i\) since \(n_i\) lies over \(m\). Then \(1 - x^2 \in B^*\) and hence \(x = 0\).

(2) Since \(\overline{B}\) is a finite dimensional \(k\)-algebra, \(\overline{B}\) is artinian. Then \(\overline{B} \simeq \prod_{i=1}^k B_{n_i}^\ast\).

\[\text{[Bou89, Ch. IV, § 2, no. 5, Prop. 9, Cor. 1].}\]

Suppose \(b_i \in \overline{B}\) such that the \(i\)-th entry of \(b_i\) is 1 and all other entries are 0. Then \(b_i \in \text{Idem}(\overline{B})\). Since local rings only have idempotents 0, 1 (otherwise there exists \(e \not\in \{0, 1\}, e(e - 1) = 0\), then \(e, e - 1 \in m, 1 \in m\), a contradiction), any idempotent of \(\overline{B}\) is a sum of some \(b_i\).

Suppose \(\text{Idem}(B) \rightarrow \text{Idem}(\overline{B})\) is surjective. Then there exists \(e_i \in \text{Idem}(B)\) such that \(\overline{e_i} = b_i\) and \(e_i\) is not the sum of other nontrivial idempotents. Hence \(e_i, e_j = 0\), \(B = \prod_{i=1}^k B_{ei}\). Since \(e_i^2 = e_i, B_{ei}\) is a ring with identity element \(e_i\). Since \(B_{ei} \simeq \overline{B_{n_i}}^\ast\) is local, there is only one prime ideal of \(B_{ei}\) containing \(mB_{ei}\). Therefore \(B_{ei}\) is a local ring for all 1 \(\leq i \leq k\).

Conversely, if \(B\) is a product of local rings, then by Proposition 1.9, \(B \simeq \prod_{i=1}^k B_{n_i}\).

Suppose \(e_i \in B\) such that the \(i\)-th entry of \(b_i\) is 1 and all other entries are 0. Then \(b_i \in \text{Idem}(\overline{B})\). Then \(\overline{e_i} = b_i\). Any idempotent of \(\overline{B}\) has inverse image the sum of some \(e_i\). Hence \(\text{Idem}(B) \rightarrow \text{Idem}(\overline{B})\) is surjective. \(\square\)
2. April 13th Product of local rings

2.1 Proposition
Let \( (A, m, k) \) be a local ring. Then every finite \( A \)-algebra is a product of local rings iff for all monic polynomial \( f(T) \in A[T] \), \( \frac{A[T]}{(f(T))} \) is a product of local rings.

Proof. Suppose for all monic polynomial \( f(T) \in A[T] \), \( \frac{A[T]}{(f(T))} \) is a product of local rings. Suppose \( B \) is a finite \( A \)-algebra. We want to show that \( \text{Idem}(B) \to \text{Idem}(\overline{B}) \) is surjective. It suffices to find inverse images for \( b_1, \ldots, b_k \) as in Proposition 1.10.

Let \( e_1 \in B \) such that \( \overline{e_1} = b_1 \). Since \( B \) is finite over \( A \), \( A[e_1] \) is finite over \( A \) and \( e_1 \) is integral over \( A \). There exists a minimal monic polynomial \( f(T) \in A[T] \) such that \( f(e_1) = 0 \). We have an \( A \)-homomorphism \( \varphi : \frac{A[T]}{f(T)} \to B \), \( \varphi(t) = e_1 \) where \( t \) is the image of \( T \) modulo \( f(T) \). Then \( \overline{\varphi} : \frac{k[T]}{(f(T))} \to A[e_1] = k[b_1] \) is surjective. The minimal polynomial \( g(T) \) of \( b_1 \) is an irreducible factor of \( f(T) \). We have

\[
\frac{k[T]}{(f(T))} \xrightarrow{\varphi} k[b_1] \\
\downarrow \quad \psi \\
\frac{k[T]}{(g(T))}
\]

where \( q \) is the quotient map and \( \psi \) is an isomorphism. Suppose \( c \in \text{Idem} \left( \frac{k[T]}{(g(T))} \right) \) such that \( \psi(c) = b_1 \). Suppose \( c \) is the image of \( c(T) \in k[T] \) modulo \( f(T) \) such that \( g(T)c(T)^n \) for some integer \( n > 0 \). Suppose \( f(T) = g(T)^a h(T) \) where \( a > 0 \) is an integer and \( g, h \) are coprime. Then \( f(T)|(c(T)h(T))^\alpha \). Let \( c' \) be the image of \( c(T)h(T) \) modulo \( f(T) \) in \( \text{Idem} \left( \frac{k[T]}{(f(T))} \right) \).

By assumption, \( \frac{A[T]}{(f(T))} \) is a product of local rings. By Proposition 1.10(2),

\[
\text{Idem} \left( \frac{A[T]}{(f(T))} \right) \to \text{Idem} \left( \frac{k[T]}{(f(T))} \right)
\]

is surjective. Let \( e \in \text{Idem} \left( \frac{A[T]}{(f(T))} \right) \to c' \).

Since \( q(t) \) is the image of \( T \) modulo \( g(T) \) and \( h(T) \equiv 1 \mod g(T) \), we have \( q(c') = c \). Therefore \( \varphi(e) = \overline{\varphi}(\varphi) = \overline{\psi}(c') = \psi(q(c')) = \psi(c) = b_1 \), we have that \( \varphi(e) \) is an inverse image of \( b_1 \) under \( \text{Idem}(B) \to \text{Idem}(\overline{B}) \). In the same manner, we can find inverse images of \( b_2, \ldots, b_k \).

Conversely, since \( \frac{A[T]}{(f(T))} \) is a finite \( A \)-algebra, it is a product of local rings. \( \square \)
2.2 Lemma
Let \((A, m, k)\) be a local ring. Let \(C\) be a finite \(A\)-algebra such that \(C \simeq \frac{k[t]}{(p_0(T))}\) for some polynomial \(p_0 \in k[\bar{T}]\) of minimal degree \(n\). Let \(\bar{t}\) be the canonical image of \(t\) in \(\frac{k[\bar{T}]}{(p_0(\bar{T}))}\). Let \(\bar{t}\) be the lift of \(\bar{t}\) to \(C\). Then there exists a monic polynomial \(p \in A[\bar{T}]\) of degree \(n\) such that \(\bar{p} = p_0\) and \(p(t) = 0\).

**Proof.** Since \((1, \bar{t}, \ldots, \bar{t}^{n-1})\) generate \(C\), by Nakayama lemma\(^1\), \((1, t, \ldots, t^{n-1})\) generate \(C\). Suppose \(t^n = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}, a_i \in A\). Let \(p(T) = T^n - a_{n-1} T^{n-1} - \cdots - a_1 T - a_0 \in A[\bar{T}]\). Then \(p(t) = 0\).

Further, \(\bar{p}(\bar{t}) = 0\) has degree \(n\). Since \(p_0\) has minimal degree, \(\bar{p} = p_0\). \(\square\)

2.3 Theorem
Let \((A, m, k)\) be a local ring. The following are equivalent:

1. For all \(f(T) \in A[T]\) monic, \(B = \frac{A[T]}{(f(T))}\) is a product of local rings.

2. If \(f \in A[T]\) such that \(\bar{f} = g h_0 \) and \(g_0, h_0\) are coprime in \(k[T]\), there exists \(g, h \in A[T]\) such that \(f = gh\), \(\bar{g} = g_0\) and \(\bar{h} = h_0\).

**Proof.** (1) implies (2). By Chinese Remainder Theorem, \(B = \frac{k[T]}{(f(T))} \simeq \frac{k[T]}{(g_0(T))} \times \frac{k[T]}{(h_0(T))}\). Since \(B\) is a product of local rings, \(\text{Idem}(B) \to \text{Idem}(\bar{B})\) is bijective.

Then there exist \(B_1, B_2\) such that \(B \simeq B_1 \times B_2\), \(B_1 = \frac{k[T]}{(g_0(T))}\) and \(B_2 = \frac{k[T]}{(h_0(T))}\).

Let \(\bar{t}\) be the canonical image of \(t\) in \(\bar{B} = \frac{k[T]}{(f(T))}\). Let \(\bar{t}\) be a lift of \(\bar{t}\) to \(B\). Then \(f(T)\) is the minimal polynomial of \(t\).

Suppose \(t = (t_1, t_2)\), \(t_1 \in B_1\) and \(t_2 \in B_2\). By **Lemma 2.2**, there exists a monic \(g \in A[T]\) such that \(g(t_1) = 0\) and \(\bar{g} = g_0\); there exists a monic \(h \in A[T]\) such that \(h(t_2) = 0\) and \(\bar{h} = h_0\). Hence \((gh)(t_1) = g(t_1)h(t_1) = (g(t_1))(h(t_1), g(t_2))h(t_2) = (0 + h(t_1), g(t_2) + 0) = 0\). Since \(\text{deg}(g) = \text{deg}(g_0), \text{deg}(h) = \text{deg}(h_0)\), we have \(\text{deg}(gh) = \text{deg}(g) + \text{deg}(h) = \text{deg}(g_0) + \text{deg}(h_0) = \text{deg}(g_0 h_0) = \text{deg}(f)\) and hence \(f = gh\).

(2) implies (1). Since \(k[T]\) is a UFD, let \(\bar{f} = \prod g_i^{e_i}\) be the decomposition into irreducible monic polynomials. By assumption, there exists a decomposition \(f = \prod f_i\) into monic polynomials in \(A[T]\) such that \(\bar{f}_i = g_i^{e_i}\). Let \(u : \frac{A[T]}{(f(T))} \to \prod \frac{A[T]}{(f_i(T))}\) be the canonical morphism. By Chinese Remainder Theorem, \(u\) is an isomorphism. By Nakayama lemma, \(u\) is surjective. Since \(\frac{A[T]}{(f(T))}, \frac{A[T]}{(f_i(T))}\) are free \(A\)-modules of finite rank and \(\det(\pi) \neq 0\), \(\det(u) \notin m\). Hence \(\det(u) \in A^*\) and \(u\) is injective. We obtain that \(u\) is an isomorphism.

\(^1\)Nakayama: Let \(R\) be a commutative ring with Jacobson ideal \(J\). Let \(M\) be a finitely generated \(R\)-module, if \(\bar{m}_1, \ldots, \bar{m}_n\) generate \(M/JM\), then \(m_1, \ldots, m_n\) generate \(M\).
If \( k[T] / (g_i(T)_{\xi_i}) \) is not a local ring, then it is a direct sum of proper ideals, which corresponds coprime factors of \( g_i(T)_{\xi_i} \), a contradiction. Since \( A[T] / (f_i(T)) = k[T] / (g_i(T)_{\xi_i}) \) is local and maximal ideals of \( A[T] / (f_i(T)) \) must contain the inverse image of the maximal ideal of \( k[T] / (g_i(T)_{\xi_i}) \), \( A[T] / (f_i(T)) \) is a local ring for all \( i \). \( \square \)
3. May 18th Equivalence conditions

3.1 Review
Let \((A, m, k)\) be a local ring. Let \(B\) be a finite \(A\)-algebra. The following are equivalent:

(a) \(B\) is isomorphic to a product of local rings.

(b) \(B \cong \prod_{i=1}^{k} B_{n_i}\) where \(n_i\) are maximal ideals of \(B\) lying over \(m\).

(c) \(\text{Idem}(B) \rightarrow \text{Idem}(\overline{B})\) is surjective.

(a) \iff (b) is proved in Proposition 1.9.

(b) \iff (c) is proved in Proposition 1.10.

3.2 Review
Let \((A, m, k)\) be a local ring. The following are equivalent:

(1) Every finite \(A\)-algebra \(B\) satisfies (a)(b)(c).

(1') Every finite free \(A\)-algebra \(B\) satisfies (a)(b)(c).

(2) For all monic \(f(T) \in A[T]\), \(\frac{A[T]}{f(T)}\) is a product of local rings.

(3) If \(f(T) \in A[T]\) such that \(\overline{f} = g_0 h_0\) for coprime \(g_0, h_0 \in k[T]\), then there exists \(g, h \in A[T]\) such that \(f = gh, \overline{g} = g_0\) and \(\overline{h} = h_0\).

Proof. 
(1) \iff (2) is proved in Proposition 2.1.

(2) \iff (3) is proved in Theorem 2.3.

(1) \implies (1') is trivial.

(1') \implies (2) since \(\frac{A[T]}{f(T)}\) is a free \(A\)-module and (1) \implies (2). \(\square\)

3.3 Lemma
Let \(\text{Clopen}(\bullet)\) be the set of closed and open subsets of a topological space. For all commutative ring \(R\), we have a bijection \(\text{Idem}(R) \simeq \text{Clopen}(\text{Spec}(R))\).

Proof. Define \(D : \text{Idem}(R) \rightarrow \text{Clopen}(\text{Spec}(R))\), where \(D(e) = \{p \in \text{Spec}(R) \mid a \not\in p\}\) is a basic open set in \(\text{Spec}(R)\).

First, we show that \(D\) is well-defined. Since \(e^2 = e, e(1 - e) = 0\). For all prime ideal \(p, e(1 - e) = 0 \in p\), then \(e \in p\) or \(1 - e \in p\), i.e. \(D(e) \cap D(1 - e) = \emptyset\). Also, \(1 = e + (1 - e) \not\in p\). Then either \(e \not\in p\) or \(1 - e \not\in p\), i.e. \(D(e) \cup D(1 - e) = \text{Spec}(R)\). Thus \(D(e) = \text{Spec}(R) \setminus D(1 - e)\) is closed.

Injectivity. Suppose \(e_1, e_2 \in \text{Idem}(R)\) such that \(D(e_1) = D(e_2)\) we denote \(U = D(e_1)\). Then \(\text{Spec}(R) \setminus U = D(1 - e_1) = D(1 - e_2)\). Since \(\text{Idem}(R_p) = \{0, 1\}\) for all \(p\), we have \(e_i \mod p = 1\) iff \(1 - e_i \in p\) iff \(e_i \not\in p\) iff \(p \in U\). Similarly, \(e_i \mod p = 0\) iff \(p \in \text{Spec}(R) \setminus U\). Thus \(e_1|_U = e_2|_U = 1\), \(e_1|_{\text{Spec}(R) \setminus U} = e_2|_{\text{Spec}(R) \setminus U} = 0\). Since \(\mathcal{O}_{\text{Spec}(R)}\) is a sheaf, \(e_1 = e_2 \in \Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) \simeq R\).

Surjectivity. Let \(U\) be any open and closed subset of \(\text{Spec}(R)\). Take \(e \in R \simeq \Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})\) such that \(e|_U = 1\) and \(e|_{\text{Spec}(R) \setminus U} = 0\). Then \(e^2 = e\). Also \(p \in U\) iff \(e \equiv 1 \mod p\) iff \(1 - e \in p\) iff \(p = \text{Spec}(R) \setminus D(1 - e) = D(e)\). Thus \(D(e) = U\). \(\square\)

3.4 Lemma
Let \(B\) be a finite free \(A\)-algebra. There exists an \(A\)-algebra \(E\) such that

(1) \(\text{Hom}_A(E, C) \simeq \text{Idem}(B \otimes_A C)\) is a functorial isomorphism for all \(A\)-algebra \(C\).
(2) $E$ is étale over $A$.

**Proof.** (1) Let $e_1, \ldots, e_n$ be a basis of $B$ such that $e_i e_j = \sum_{k=1}^{n} a_{i,j,k} e_k$. If $b = \sum_{i=1}^{n} b_i e_i \in B$ such that $b^2 = b$, then $b_k = \sum_{i,j=1}^{n} b_i b_j a_{i,j,k}$. Let

$$P_k(T_1, \ldots, T_n) = \sum_{i,j=1}^{n} a_{i,j,k} T_i T_j - T_k, \quad 1 \leq k \leq n$$

be polynomials of $A[T_1, \ldots, T_n]$ of degree $\leq 2$. Let $E = \frac{A[T_1, \ldots, T_n]}{(P_1, \ldots, P_n)}$. Let $t_i$ be the canonical image of $T_i$ in $E$. Define

$$\varphi : \text{Hom}_A(E, C) \to \text{Idem}(B \otimes_A C), \quad \varphi(u) = \sum_{i=1}^{n} e_i \otimes u(t_i).$$

**Functoriality.** Suppose $f : C \to C'$ is an $A$-homomorphism. Then $f \circ u \in \text{Hom}_A(E, C')$. Thus $\varphi(f \circ u) = \sum_{i=1}^{n} e_i \otimes f(u(t_i)) = (1 \otimes f)(\varphi(u))$.

**Injectivity.** $\varphi(u) = 0$ iff $u(t_i) = 0$ for all $1 \leq i \leq n$ iff $u = 0$.

**Surjectivity.** Since $e_i \otimes 1$ is a basis of $B \otimes_A C$, an element $\sum_{i=1}^{n} e_i \otimes c_i, \quad c_i \in C$ is idempotent in $B \otimes_A C$ iff $P_k(e_1, \ldots, e_n) = 0$ for all $1 \leq k \leq n$. This yields an $A$-homomorphism $u : E \to C$ with $u(t_i) = c_i$.

(2) By construction, $E$ is of finite presentation over $A$. It suffices to show that the canonical map $\text{Hom}_A(E, C) \to \text{Hom}_A(E, C/J)$ is a bijection for all $A$-algebra $C$ with an ideal $J$ such that $J^2 = 0$. By (1), it is enough to show that $\text{Idem}(B \otimes_A C) \to \text{Idem}(B \otimes_A C/J)$ is a bijection. By Lemma 3.3, it is enough to show that $\text{Clopen}(\text{Spec}(B \otimes_A C)) \to \text{Clopen}(\text{Spec}(B \otimes_A C/J))$ is a bijection, which is true since $\text{Spec}(B \otimes_A C) \simeq \text{Spec}(B \otimes_A C/J)$ as $B \otimes_A J$ is nilpotent (All nilpotent elements are contained in every prime ideal).

$\square$

### 3.5 Theorem

Conditions of Review 3.2 are also equivalent to:

(4) $A$ is henselian.

(5) Let $B$ be an étale over $A$ with a maximal ideal $n$ lying over $m$ such that $B/n \simeq k$. Then the structure morphism $A \to B$ provides an isomorphism $A \simeq B_n$.

(6) For all étale morphism $g : X \to \text{Spec}(A)$ and $\overline{g} : X \times_{\text{Spec}(A)} \text{Spec}(k) \to \text{Spec}(k)$, if $s_0$ is a section of $\overline{g}$, then there exists a unique section $s$ of $g$ such that $\overline{s} = s_0$.

**Proof.** (3) implies (4). Suppose $f \in A[T]$ is monic such that $f(a_0) = 0$ and $\overline{f}(a_0) \neq 0$. Then $\overline{f} = g_0 h_0$ where $g_0(T) = T - a_0$ and $h_0(a_0) \neq 0$. By (3), we have $f = gh$ for some monic $g, h \in A[T]$ such that $\overline{g} = g_0$ and $\overline{h} = h_0$. Since $g$ is monic, $g(T) = T - a$ such that $\overline{a} = a_0$ and $f(a) = 0$.

(4) implies (5). Since $B$ is étale over $A$. Then there exists $b \in B \setminus n$ such that $B_b$ is standard étale over $A$. Suppose $B_b = \frac{A[T]}{(f(T))}$ for some monic $f \in A[T]$.
Since
\[ \text{Spec}(B_b) \simeq \{ p \in \text{Spec}(A[T]) \mid f'(T) \notin p \} \]
\[ \simeq \{ p \in \text{Spec}(A[T]) \mid f(T) \in p, \ f'(T) \notin p \} \]
we have
\[ \text{Spec}(B_b/m) \simeq \{ p \in \text{Spec}(A[T]) \mid f(T) \in p, \ f'(T) \notin p, \ p \supset m \} \]
\[ \simeq \{ p \in \text{Spec}(k[T]) \mid f(T) \in p, \ f'(T) \notin p \}. \]
Since \( B/n \simeq k \) and \( b \notin n \), we have \( B_b/n_b \simeq k_b = k \). The maximal ideal \( n \) of \( B_b \) corresponds to a maximal ideal \( T - a_0 \) of \( k[T] \) for some \( a_0 \in k \) such that \( f(a_0) = 0 \) and \( f'(a_0) \neq 0 \). By (6), there exists \( a \in A \) such that \( \overline{a} = a_0 \) and \( f(a) = 0 \). Then \( f(T) = (T - a)g(T) \) for some \( g(T) \in A[T] \). Similar to Theorem 2.3, the following canonical morphism is an isomorphism.
\[ \varphi : \frac{A[T]}{(f(T))} \rightarrow \frac{A[T]}{(T - a)} \times \frac{A[T]}{(g(T))} = A \times \frac{A[T]}{(g(T))} \]

Similar to Proposition 1.9, \( A \simeq \left( \frac{A[T]}{(f(T))} \right)_n \).

Since \( f'(T) \notin p \) for all \( p \in \text{Spec}(B_b) \), \( f'(T) \notin n_b = n \).

Since \( b \notin n \), \( (B_b)_n \simeq B_n \).

Therefore \( A \simeq \left( \frac{A[T]}{(f(T))} \right)_n \simeq \left( \frac{A[T]}{(f(T))} \right)^b = (B_b)_n \simeq B_n \).

(5) implies (1')(c). Suppose \( B \) is a finite free \( A \)-algebra with basis \( b_1, \ldots, b_n \). We want to show that \( \text{Idem}(B) \rightarrow \text{Idem}(\overline{B}) \) is surjective. Let \( E \) be as in Lemma 3.4. Then
\[ \text{Hom}_A(E, k) \simeq \text{Idem}(B \otimes_A k) = \text{Idem}(\overline{B}). \]

Let \( e_0 = \sum_{i=1}^{n} c_i b_i \in \text{Idem}(\overline{B}) \) which corresponds to an \( A \)-homomorphism \( u_0 : E \rightarrow k \) such that \( u_0(a) = \overline{a} \) for all \( a \in A \) and \( u_0(t_i) = c_i \in k \). Let \( n = \ker(u_0) \). Then we have a commutative diagram
\[ \begin{array}{ccc}
E & \xrightarrow{u_0} & k \\
\downarrow{i} & & \downarrow{v_0} \\
E_n & \xrightarrow{v_0} & k
\end{array} \]

where \( v_0(x) = \frac{u_0(x)}{v_0(x)} \) for all \( x \in E \) and \( s \in E \setminus n \).

By Lemma 3.4, the \( A \)-algebra \( E \) has an étale structure morphism \( j : A \rightarrow E \). Since \( m \subset n \), \( E/n \simeq k \), \( n \) is a maximal ideal of \( E \). By (5), the composition \( i \circ j : A \rightarrow E_n \) is an isomorphism. Let \( v = (i \circ j)^{-1} \) and \( u = v \circ i \). Since \( E_n/nE_n \simeq E/n \simeq k \). We have \( \overline{v} = v_0 \) and hence \( \overline{u} = v_0 \circ i = u_0 \). By Lemma 3.4
\[ \text{Hom}_A(E, A) \simeq \text{Idem}(B \otimes_A A) = \text{Idem}(B). \]
Then $u$ corresponds $e = \sum_{i=1}^{n} u(t_i) b_i \in \text{Idem}(B)$ such that $e = \sum_{i=1}^{n} u_0(t_i) b_i = \sum_{i=1}^{n} c_i b_i = e_0$.

Therefore (1)(2)(3)(4)(5) are equivalent.

(6) implies (1)(c). Since $g : \text{Spec}(E) \to \text{Spec}(A)$ is étale. By (6)

$$\text{Hom}_{\text{Spec}(A)}(\text{Spec}(A), \text{Spec}(E)) \to \text{Hom}_{\text{Spec}(k)}(\text{Spec}(k), \text{Spec}(E) \times_{\text{Spec}(A)} \text{Spec}(k))$$

is surjective. Then $\text{Hom}_A(E, A) \to \text{Hom}_k(E \otimes_A k, k)$ is surjective. By Lemma 3.4, $\text{Idem}(B) \to \text{Idem}(B)$ is surjective. The above surjective homomorphisms are all bijective.

$(4) \implies (6)$ will be proved next time. □

Main reference [Fu15].

3.6 Lemma

Let $(A, m, k)$ be a local ring. Let $B$ be a finite $A$-algebra with maximal ideals $n_1 = n, n_2, \ldots, n_k$. Suppose $t \in \bigcap_{i=2}^{k} n_i \setminus n$ and $\mathfrak{t}$ generates $B_n \otimes_A k$ as an $k$-algebra.

Then the canonical morphism $A[t]_{n \cap A[t]} \to B_n$ is an isomorphism.

Proof. First, we show that $B_{n \cap A[t]} \simeq B_n$.

Since $B$ is finite over $A$, $B_{n \cap A[t]}$ is finite over $A[t]_{n \cap A[t]}$. Any maximal ideal $n'_{n \cap A[t]}$ of $B_{n \cap A[t]}$ lies over the maximal ideal $n \cap A[t]$ of $A[t]$, where $n'$ is a maximal ideal of $B$. If $t' \in n'$, then $t \in n' \cap A[t] = n \cap A[t] \subset n$, a contradiction to our assumption.

So $t \notin n'$, by assumption $n' = n$. Hence $B_{n \cap A[t]}$ is a local ring with maximal ideal $n_{n \cap A[t]}$. Since the image of every element of $B \setminus n$ in $B_{n \cap A[t]}$ is invertible, we have $B_n \to B_{n \cap A[t]}$. Since $A[t] \setminus (n \cap A[t]) \subset B \setminus n$, we have $B_{n \cap A[t]} \to B_n$.

Now we prove that the canonical morphism $A[t]_{n \cap A[t]} \to B_n$ is an isomorphism.

Injectivity. Since $A[t] \subset B$, $A[t]_{n \cap A[t]} \to B_{n \cap A[t]} \simeq B_n$ is injective.

Surjectivity. Since $\mathfrak{t}$ generates $B_n$, the canonical morphism $A[t]_{n \cap A[t]} \to B_n$ is an isomorphism. By Nakayama lemma, $A[t]_{n \cap A[t]} \to B_n$ is surjective. □

3.7 Theorem

Chevalley. Let $(A, m, k)$ be a local ring. Let $X$ be a scheme with a morphism $g : X \to \text{Spec}(A)$ étale at $x$ and $g(x) = m$. Then $\mathcal{O}_{X,x} \simeq \frac{A[T]}{(F(T))_p}$ for some monic polynomial $F(T) \in A[T]$ such that $F' \notin p$.

Proof. Since $g$ is étale at $x$, there exists an affine open neighborhood $\text{Spec}(C)$ of $x$ such that $C$ is (standard) étale over $A$. Then $C$ is quasi-finite over $A$. Since $g(x) = m$, $x$ is a maximal ideal of $C$ lying over $m$. By a corollary of Zariski Main theorem, there exists a finite $A$-algebra $B \subset C$ such that $\text{Spec}(C) \to \text{Spec}(B)$ is a homeomorphism onto its image and the image is open. Let $n$ be the image of $x$.

Then $\mathcal{O}_{X,x} \simeq C_x \simeq B_n$.

Suppose $n_1 = n, n_2, \ldots, n_k$ are distinct maximal ideals of $B$ lying over $m$. Since $g$ is étale at $n$, $B/n$ is a finite separable extension of $k$. By primitive element theorem,
$B/n = k[\overline{t}]$ for some $\overline{t} \in (B/n) \setminus k$. By Chinese Remainder theorem, there is a surjection

$$B \to B/\prod_{i=1}^{k} n_i \cong (B/n) \times (B/n_2) \times \cdots \times (B/n_k).$$

Let $t \in B$ with image $(\overline{t}, 0, \ldots, 0)$. Then $t \in \prod_{i=2}^{k} n_i \setminus n$. By Lemma 3.6, $A[t]_{n \cap A[t]} \cong B_n$.

Suppose $k[\overline{t}] = \frac{k[T]}{(F(T))}$ for some $F(T) \in A[T]$ such that $F(t) = 0$ and $F(T)$ is the minimal polynomial of $\overline{t}$. Since $k[\overline{t}]$ is finite separable over $k$, $F$ has no multiple roots. Hence $F'(t) \not\in (n \cap A[t]) \otimes_A k$, $F'(t) \not\in n \cap A[t]$. We have a surjection $A[T] \to A[t]$ defined by $F(T) = t$. Then the image of $F'(T)$ in $A[T]$ is not contained in $\phi^{-1}(n)$. Then $A \to \frac{A[T]}{(F(T))}\phi^{-1}(n \cap A[t])$ is (standard) étale.

Now, we show that $\phi': \frac{A[T]}{(F(T))}\phi^{-1}(n \cap A[t]) \to A[t]_{n \cap A[t]}$ is an isomorphism.

**Surjectivity** follows from the fact that $\phi$ is surjective.

**Injectivity.** Since the composition $A \to \frac{A[T]}{(F(T))}\phi^{-1}(n \cap A[t]) \to A[t]_{n \cap A[t]}$ is étale, the second map $\phi'$ is étale [Tam94, 1.1.2(iv)] and hence flat. Since $\phi'$ is a local homomorphism of local rings and $A[t]_{n \cap A[t]} \otimes_A k = k[\overline{t}] \neq 0$. Hence $\phi'$ is faithfully flat. Since $\phi'$ is faithfully flat, it is injective.

Finally, let $p = \phi^{-1}(n) \cap A[t]$, we have $\mathcal{O}_{X,x} \cong \frac{A[T]}{(F(T))}_p$. \hfill \qed

---

2Étale=flat+unramified.

3An $R$-module $M$ is faithfully flat if as $R$-modules and $R$-morphisms

$$N_1 \to N_2 \to N_3$$

is exact $\iff N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact.

An $R$-module $M$ is faithfully flat iff $M \otimes_R R/m \neq 0$ for all maximal $m$ of $R$.

4Any faithfully flat algebra homomorphism $f : R \to S$ is injective. Proof: $\ker(f) \to R \xrightarrow{f} S$ is exact iff $0 \to S \to S \otimes_R S$ is exact iff $0 \to R \xrightarrow{f} S$ is exact.
4. June 1st Henselization

4.1 Lemma

Let \((A, \mathfrak{m}, k)\) be a local ring. Suppose \(B\) is an \(A\)-algebra of finite type. Suppose \(B\) is unramified at a maximal ideal \(\mathfrak{n}\) lying over \(\mathfrak{m}\). Suppose \(A'\) is a local \(A\)-algebra with maximal ideal \(\mathfrak{m}'\) lying over \(\mathfrak{m}\). Then there exists a unique section

\[ s : \text{Spec}(A') \to \text{Spec}(B \otimes_A A') \simeq \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(A') \]

of the projection \(\text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(A') \to \text{Spec}(A')\) and \(s(\mathfrak{m}') = (\mathfrak{n}, \mathfrak{m}')\).

**Proof.** Since \(B\) is unramified at a maximal ideal \(\mathfrak{n}\) lying over \(\mathfrak{m}\), \(k' = \frac{B_n}{\mathfrak{n}B_n}\) is a finite separable extension of \(k\). For all \(t \in k'\), let \(f(T) \in k[T]\) be its minimal polynomial. Then \(f(t) = 0\) and \(f'(t) \neq 0\).

We have a Cartesian diagram

\[
\begin{array}{ccc}
B_n \otimes_A B_n & \xrightarrow{\mu} & B_n \\
\downarrow{\iota \circ \mu} & & \downarrow{f_{\circ i}} \\
B_n \otimes_A A' & \xrightarrow{f} & A'
\end{array}
\]

where \(\mu\) is the multiplication \(\mu(b_1 \otimes b_2) = b_1 b_2\) for all \(b_1, b_2 \in B_n\); \(i(b) = b \otimes 1\) for all \(b \in B_n\); \(A' \simeq B_n \otimes (B_n \otimes A) (B_n \otimes A)'\); and \(f\) is given by \(f(x) = 1 \otimes x\).

Let \(d : k' \to k' \otimes_k k'\) such that \(d(x) = x \otimes 1 - 1 \otimes x\) for all \(x \in k'\). Then \(d\) is a derivation. Since \(0 = d(f(t)) = f'(t)d(t)\), we have \(d(t) = 0\) for all \(t \in k'\), i.e. \(\text{im}(d) = 0\).

For all \(\sum_i x_i \otimes y_i \in \ker(\mu)\), \(\sum_i x_i y_i = 0\). Then

\[
\sum_i x_i \otimes y_i = \sum_i x_i \otimes y_i - \left(\sum_i x_i y_i\right) \otimes 1 = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).
\]

Suppose \(B_n = A[b_1, \ldots, b_n]\). Then \(\ker(\mu)\) is finitely generated by \(b_i \otimes 1 - 1 \otimes b_i\), \(1 \leq i \leq n\). Then \(\ker(\mu) \otimes k'\) is generated by \(\text{im}(d)\), \(\ker(\mu) \otimes k' = 0\). By Nakayama lemma, \(\ker(\mu) = 0\). Hence \(\mu'\) is an isomorphism. By the Cartesian diagram, \(f\) is an isomorphism.

For any section \(s : \text{Spec}(A') \to \text{Spec}(B \otimes_A A')\) such that \(s(\mathfrak{m}') = (\mathfrak{n}, \mathfrak{m}')\), we have a section \(s : \text{Spec}(A') \to \text{Spec}(B_n \otimes_A A')\). Then \(f : B_n \otimes_A A' \to A'\) is identified with

\(- \circ s : \text{Hom}(\text{Spec}(B_n \otimes_A A'), \text{Spec}(A[T])) \to \text{Hom}(\text{Spec}(A'), \text{Spec}(A[T]))\)

and \(s\) is identified with \(\text{Spec}(f)\).

\(\square\)

(4) implies (6). Let \(A\) be a henselian local ring. We have \(g : X \to \text{Spec}(A)\). Suppose \(s_0 : \text{Spec}(k) \to X \times_{\text{Spec}(A)} \text{Spec}(k)\), \(s_0(\ast) = (x, \ast)\) where \(\mathfrak{p}(x) = \ast\) and \(\text{Spec}(k) = \{\ast\}\). Since \(g\) is étale at \(x\), by Theorem 3.7, \(\mathcal{O}_{X,x} \simeq \frac{A[T]}{(f(T))}\) where \(\mathfrak{n}\) is a maximal ideal of \(\frac{A[T]}{(f(T))}\) not containing the image of \(f'\) and lying over \(\mathfrak{m}\). We obtain a canonical morphism

\[
\frac{k[T]}{(f(T))} \otimes_A k \to \frac{A[T]}{(f(T))}_n \otimes_A k \simeq \mathcal{O}_{X,x} \otimes_A k
\]
The section $s_0$ provides a $k$-algebra homomorphism $\mathcal{O}_{X,x} \otimes_A k \rightarrow k$. Their composition is a $k$-algebra homomorphism $\overline{\varphi} : \frac{k[T]}{\langle f(T) \rangle} \rightarrow k$.

Let $a_0 = \overline{\varphi}(T \bmod (f(T)))$. Then $\overline{f}(a_0) = 0$ and $\overline{f}'(a_0) \neq 0$. By (4), there exists a unique $a \in A$ such that $\overline{a} = a_0$ and $f(a) = 0$. Then we have an $A$-homomorphism $\varphi : \frac{A[T]}{\langle f(T) \rangle} \rightarrow A$ defined by $\varphi(T \bmod f(T)) = a$. Then $\varphi$ defines an $A$-homomorphism $\varphi' : \frac{A[T]}{\langle f(T) \rangle} \rightarrow A$. We call the following composition $s$

$$\text{Spec}(A) \xrightarrow{\text{Spec}(\varphi')} \text{Spec} \left( \frac{A[T]}{\langle f(T) \rangle} \right) \simeq \text{Spec} (\mathcal{O}_{X,x}) \rightarrow X$$

Let $\text{Spec}(B)$ an affine open neighborhood of $x$. The canonical map $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ is given by $B \mapsto B_x \simeq \mathcal{O}_{X,x}$ and is independent of the choice of $B$.

Finally, $s$ is a section of $g$ since $\varphi'$ is an $A$-homomorphism. Also, $\overline{s} = s_0$ since $\overline{\varphi} = \overline{s}$ and $s_0$ is the composition

$$\text{Spec}(k) \xrightarrow{\text{Spec}(\overline{s})} \text{Spec}(\mathcal{O}_{X,x} \otimes_A k) \rightarrow X \times_{\text{Spec}(A)} \text{Spec}(k).$$

Since $s(m) = (x, m)$, by Lemma 4.1, $s$ is unique.

### 4.2 Definition

Let $(A, m, k)$ be a local ring. An $A$-algebra $C$ is **essentially étale** if there exists an étale $A$-algebra $B$ with a prime ideal $n$ over $m$ such that $C \simeq B_n$.

### 4.3 Lemma

If $A'$ is essentially étale over $A$ and $A''$ is essentially étale over $A''$, then $A''$ is essentially étale over $A$.

**Proof.** Suppose $A' \simeq B_n$ where $B$ is étale over $A$ and $n$ lies over $m$. Suppose $A'' \simeq B'_n$, where $B'$ is étale over $A'$ and $n'$ lies over $nB_n$. Since $B'$ is étale over $A' = B_n$, there exists $h' \in B' \setminus n'$ such that $B'_n$, standard étale over $B_h$. Since $B_h$ is étale over $B$ and $B$ is étale over $A$, by transitivity, $B'_h$, is étale over $A$. Since $h' \not\in n'$, $A'' \simeq B''_n \simeq (B'_h)_n$ is essentially étale over $A$. \hfill $\square$

### 4.4 Lemma

Suppose $\varphi : A \rightarrow A'$ is essentially étale. Let $k'$ be the residue field of $A'$. Suppose $k' \simeq k$. For all henselian local ring $H$, the following map is bijective

$$\bullet \circ \varphi : \text{Hom}_{\text{loc}}(A', H) \rightarrow \text{Hom}_{\text{loc}}(A, H)$$

**Proof.** Suppose $(H, m_H, k_H)$ is the henselian local ring. Suppose $A' \simeq B_n$ where $\psi : A \rightarrow B$ is étale, $\psi^{-1}(n) = m$ and $\varphi = i \circ \psi$, where $i : B \rightarrow B_n = A'$ is the canonical map.

Suppose $p : A \rightarrow H$ is a local homomorphism. Then $H$ becomes an $A$-algebra and $m_H$ lies over $m$. Consider the projection

$$p_2 : \text{Spec}(B \otimes_A H) \simeq \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(H) \rightarrow \text{Spec}(H)$$

Its reduction modulo $m$ is still the projection

$$\overline{p_2} : \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(k_H) \rightarrow \text{Spec}(k_H)$$
Suppose \( s_0 : \text{Spec}(k_H) \to \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(k_H) \) is the section of \( \overline{p} \) such that \( s_0(s) = (n,*). \) Since \( H \) is henselian, by (4) \( \iff \) (6), there exists a unique section \( s : \text{Spec}(H) \to \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(H) \) of \( p \) such that \( \overline{p} = s_0. \) In particular, \( s(m_H) = (n,m_H). \)

Since \( H \) is local, \( s \) is identified with \( s' : \text{Spec}(H) \to \text{Spec}(B_n) \times_{\text{Spec}(A)} \text{Spec}(H) \) such that \( s'(p) = (r,p) \) where \( \varphi^{-1}(r) = f^{-1}(p). \) The composition of \( s' \) with the projection

\[
p_1 : \text{Spec}(B_n) \times_{\text{Spec}(A)} \text{Spec}(H) \to \text{Spec}(B_n) = \text{Spec}(A')
\]

gives \( p_1 \circ s' : \text{Spec}(H) \to \text{Spec}(A') \) over \( \text{Spec}(A) \) which corresponds to a unique local homomorphism \( f' : A' \to H \) such that \( f'^{-1}(p) = r. \) Hence \( \varphi^{-1}(r) = \varphi^{-1}(f'^{-1}(p)) = f^{-1}(p), \) \( f' \circ \varphi = f. \) The existence of \( f' \) gives the surjectivity. The uniqueness of \( f' \) gives the injectivity. \( \square \)

4.5 Definition

An \( A \)-algebra \( A' \) is strictly essentially étale if it is essentially étale with residue field \( k' \) such that \( k' \simeq k. \)

4.6 Lemma

For all strictly essentially étale \( A \)-algebras \( A_1 \) and \( A_2, \) there exists at most one local homomorphism \( A_1 \to A_2. \)

**Proof.** Suppose \( A_i = (B_i)_{n_i} \) where \( B_i \) is étale over \( A \) and \( n_i \) lies over \( m \) for \( i \in \{1, 2\}. \) Let \( i_1 : B_1 \to (B_1)_{n_1} \) be the canonical map. If there exists \( f \in \text{Hom}_A(A_1, A_2), \) then \( f_1 = i_1 \circ f \in \text{Hom}_A(B_1, A_2) \) such that \( f_1^{-1}(n_2) = n_1. \) Hence \( f_2 = 1 \otimes f_1 \in \text{Hom}_A(B_1 \otimes_A A_2, A_2) \) such that \( f_2^{-1}(n_2) = (n_1 \otimes A_2) \oplus (B_1 \otimes n_2 A_2). \) By Lemma 4.1, \( f_2 \) is unique. \( \square \)

4.7 Definition

Let \( (A_{\lambda}) \) be a set of representatives of the set of isomorphism classes of strictly essentially étale \( A \)-algebras. If there exists a local homomorphism \( \varphi_{\lambda \mu} : A_{\lambda} \to A_{\mu} \) (unique by Lemma 4.6), then we define \( \lambda \leq \mu. \) We call

\[
A^h = \lim_{\longrightarrow} A_{\lambda}
\]

the henselization of \( A. \)

4.8 Lemma

For all strictly essentially étale \( A \)-algebras \( A_1 \) and \( A_2, \) there exists a strictly essentially étale \( A \)-algebras \( A_3 \) and two local \( A \)-homomorphisms \( A_1 \to A_3 \) and \( A_2 \to A_3. \)

**Proof.** Let the residue field of \( A_1, A_2 \) be \( k_1, k_2, \) respectively. Suppose \( A_i = (B_i)_{n_i} \) where \( B_i \) is étale over \( A \) and \( n_i \) lies over \( m \) for \( i \in \{1, 2\}. \) Then

\[
\begin{array}{ccc}
\text{Spec}(B_i) & \xrightarrow{a_i} & \text{Spec}(B) \\
\downarrow{b_i} & & \downarrow{d} \\
\text{Spec}(k_i) & \xrightarrow{c_i} & \text{Spec}(k) \\
\end{array}
\]

is commutative for \( i \in \{1, 2\}. \) Since \( a_1 \circ b_1 \circ c_1^{-1} = d = a_2 \circ b_2 \circ c_2^{-1}, \) we have \( \text{Spec}(k) \to \text{Spec}(B_i) \times_{\text{Spec}(A)} \text{Spec}(B_2) \simeq \text{Spec}(B_1 \otimes_A B_2) \) defined by \( b_1 \circ c_1 \) and \( b_2 \circ c_2, \) i.e. there exists a prime ideal \( n_3 \) of \( B_3 = B_1 \otimes_A B_2 \) with residue field \( k. \) Let
\( A_3 = (B_3)_{A_3} \). Since “étale” is preserved under tensor product, \( B_3 \) is étale over \( A \) and hence \( A_3 \) is strictly essentially étale. \( \square \)

4.9 Corollary

The definition of \( A^h \) is independent of the choice of \((A_{\lambda}, \varphi_{\lambda\mu})\).

**Proof.** If \((B_{\mu})\) is another set of representatives of the set of isomorphism classes of strictly essentially étale \( A \)-algebras, by Lemma 4.8, for all pairs of indices \((\lambda, \mu)\) there exists strictly essentially étale \( A \)-algebras \( C_{\lambda\mu} \) with local homomorphisms \( A_{\lambda} \to C_{\lambda\mu} \) and \( B_{\mu} \to C_{\lambda\mu} \). Then \( \lim_{\longrightarrow} A_{\lambda} = \lim_{\longrightarrow} C_{\lambda\mu} = \lim_{\longrightarrow} B_{\mu} \). \( \square \)

4.10 Example

If \( k \) is a field, then \( k^h = k \).

**Proof.** An étale \( k \)-algebra is a finite product of finite separable field extensions of \( k \). An essentially étale \( k \)-algebra is a finite separable field extension of \( k \). A strictly essentially étale \( k \)-algebra is \( k \) itself and hence \( k^h = k \). \( \square \)
5. June 8th Reduction to the residue field

5.1 Lemma
If $S$ is a strictly essentially étale $A$-algebra, then $S^h \simeq A^h$.

Proof. Take $(A_\lambda, \varphi_{\lambda \mu})$ such that $S = A_{\lambda_0}$ and $A^h = \lim A_\lambda$. Let $B$ be a strictly essentially étale $S$-algebra. By Lemma 4.3, $B$ is an essentially étale $A$-algebra. Since the residue field of $B$ is isomorphic to the residue field of $S$ which is isomorphic to $k$. We have $B$ is a strictly essentially étale $A$-algebra. Then there exists $\mu \geq \lambda_0$ such that $B \simeq A_\mu$. Then $(A_\mu, \varphi_{\mu \lambda'}, \mu, \mu' \geq \lambda_0)$ is cofinal to $(A_\lambda, \varphi_{\lambda \lambda'})$. Therefore

$$S^h = \lim A_\mu \simeq \lim A_\lambda = A^h.\quad\square$$

5.2 Lemma
Given an inductive system of local rings $(A_\lambda, m_\lambda, k_\lambda)$ with local homomorphisms $f_{\lambda \mu} : A_\lambda \to A_\mu$. Let $A' = \lim A_\lambda$, $m' = \lim m_\lambda$, $k' = \lim k_\lambda$. Let $f_\lambda : A_\lambda \to A'$ be the canonical homomorphism.

1. $(A', m', k')$ is a local ring.
2. If $m_\mu = m_\lambda A_\mu$ for all $\lambda < \mu$, then $m' = m_\lambda A'$ for all $\lambda$.

Proof. (1) Take $x \in A' \setminus m'$. Suppose $x = f_\lambda(x_\lambda)$ for some $x_\lambda \in A_\lambda$. If $x_\lambda \in m_\lambda$, then $f_{\lambda \mu}(x_\lambda) \in m_\mu$ for all $\mu > \lambda$. Then $x \in m'$, a contradiction. Hence $x_\lambda \in A_\lambda \setminus m_\lambda = A^*$ and $f_\lambda(x_\lambda^{-1}) = x^{-1}$. Therefore $m'$ is the unique maximal ideal of $A'$.

Since $\lim$ is an exact functor, $k' = A'/m'$.

(2) Since $m_\mu = m_\lambda A_\mu$, the multiplication map $m_\lambda \otimes_{A_\lambda} A_\mu \to m_\mu$ is surjective. Since $\lim$ commutes with tensor product, $m_\lambda \otimes_{A_\lambda} A' = m_\lambda \otimes_{A_\lambda} \lim A_\mu \simeq \lim (m_\lambda \otimes_{A_\lambda} A_\mu)$. Since $\lim$ is exact, $m_\lambda \otimes_{A_\lambda} A' \simeq \lim (m_\lambda \otimes_{A_\lambda} A_\mu) \to \lim m_\mu = m'$ is surjective. Hence $m_\lambda A' = m'$ for all $\lambda$.\quad\square

5.3 Proposition
$(A^h, mA^h, k)$ is a local ring. In particular, $A \to A^h$ is local.

Proof. We have $A^h = \lim A_\lambda$ where $A_\lambda$ are strictly essentially étale $A$-algebras. Let $m_\lambda$ be the maximal ideal of $A_\lambda$. Then $A_\lambda/m_\lambda \simeq k$ and hence $m_\lambda \simeq mA_\lambda$. By Lemma 5.2, $(A^h, m', k')$ is a local ring where $m' = \lim m_\lambda \simeq \lim mA_\lambda = mA^h$ and $k' = \lim A_\lambda/m_\lambda = \lim k = k$.\quad\square

5.4 Lemma
Let $D, C$ be essentially étale $A$-algebras. Then any $A$-homomorphism $u : D \to C$ is local.

---

5 Proved by diagram chasing. Also, inverse limits are only left exact.

6 Proved by universal properties. Also, inverse limits do not necessarily commute with tensor product.
Proof. Suppose $D = B_n$ where $B$ is étale over $A$ and $n$ is a maximal ideal of $B$ lying over $m$. Let $p$ be the maximal ideal of $C$. Then $u^{-1}(p)$ is a prime ideal of $B_n$, $q = u^{-1}(p) \cap B$ is a prime ideal of $B$ lying over $m$ and $q \subseteq n$. Since $B$ is étale over $A$, $B \otimes k$ is a finite product of extensions of $k$. All prime ideals of $B \otimes k$ are maximal. Hence $q$ is maximal among prime ideals of $B$ lying over $m$, $q = n$. Hence $u^{-1}(p) = nB_n$ is the maximal ideal of $D = B_n$. \hfill $\square$

5.5 Lemma

Let $A$ be a ring. Let $u : B \to C$ be a homomorphism $A$-algebras. Let $J$ be an ideal of $C$ such that $J^2 = 0$. Then

\begin{itemize}
  \item $\ast + u : \text{Der}_A(B, J) \to \{ v \in \text{Hom}_A(B, C) \mid \pi = \pi : B \to C/J \}$
\end{itemize}

is a bijection.

Proof. The $B$-module structure of $J$ is given by $u(b)j$ for all $b \in B$ and $j \in J$.

Injectivity. For all $d_1, d_2 \in \text{Der}_A(B, J)$, if $d_1 + u = d_2 + u$, then $d_1 = d_2$.

Surjectivity. Suppose $v : B \to C$ such that $\pi = \pi$. We show that $d = v - u \in \text{Der}_A(B, J)$.

Since $\overline{d} = \overline{v} - \overline{u} = 0$, $v - u \in J$. For all $b_1, b_2 \in B$

$$d(b_1b_2) = v(b_1b_2) - u(b_1b_2),$$

where $d(b_1)d(b_2) \in J^2 = 0$. Therefore $d(b_1b_2) = b_1 \cdot d(b_2) + b_2 \cdot d(b_1)$. \hfill $\square$

5.6 Definition

Let $\mu : B \otimes_A B \to B$ be the multiplication of an $A$-algebra $B$. Let $I = \ker(\mu)$. Then $\Omega_{B/A} = I/I^2$ is called the module of relative differentials of $B$ over $A$.

We have the canonical $A$-derivation $d : B \to \Omega_{B/A}$. $d(b) = 1 \otimes b \otimes b \otimes 1 \mod I^2$.

5.7 Lemma

For all $B$-module $M$ with a derivation $d' : B \to M$, there exists a unique homomorphism of $B$-modules $f : \Omega \to M$ such that $f \circ d = d'$, i.e. a canonical bijection

\begin{itemize}
  \item $\ast \circ d : \text{Hom}_B(\Omega_{B/A}, M) \to \text{Der}_A(B, M)$
\end{itemize}

Proof. The map is well-defined since $d$ is an $A$-derivation.

Injectivity. For all $\sum x_i \otimes y_i \in I$, $\sum x_iy_i = 0$,

$$\sum x_i \otimes y_i = \sum x_i \otimes y_i - \sum x_iy_i \otimes 1 = \sum (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

If $f \circ d = 0$, then $f = 0$. Then $d$ is surjective.

Surjectivity. Given $d'$, define $f(\sum x_i \otimes y_i \mod I^2) = \sum x_i \otimes d'(y_i)$. Since

$$f((1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1)) = f((1 \otimes ab - b \otimes a - a \otimes b + ab \otimes 1) = d'(ab) - b \cdot d'(a) - a \cdot d'(b) = 0$$

Consider $\Omega_{B/A}$ as a $B$-module by left multiplication, $f$ is a $B$-homomorphism. \hfill $\square$
5.8 Definition
Let $A$ be a commutative ring. An $A$-algebra $B$ is unramified if $B$ is of finite type over $A$ and for all $A$-algebra $C$ with an ideal $J$ such that $J^2 = 0$, the canonical map $\text{Hom}_A(B, C) \rightarrow \text{Hom}_A(B, C/J)$ is a bijection.

5.9 Lemma
If $B$ is unramified over a commutative ring $A$, then $\Omega_{B/A} = 0$. (The converse of this is also true, but we do not need it at the moment.)

Proof. Let $J = \Omega_{B/A} = I/I^2$. Then $J$ is an ideal of $B \oplus J$ and $J^2 = 0$. Since $B$ is unramified over $A$, the canonical map $\text{Hom}_A(B, B \oplus J) \rightarrow \text{Hom}_A(B, B/J)$ is a bijection. There exists a unique $u: B \rightarrow B \oplus J$, $u(b) = (b, 0)$ corresponding to $\pi = \text{Id}_B: B \rightarrow \frac{B_{\oplus J}}{I} \simeq B$. By Lemma 5.5, $\text{Der}_A(B, J) = 0$. By Lemma 5.7, $\text{Hom}_B(J, J) = 0$. Therefore $J = 0$. □

5.10 Lemma
Let $R$ be a commutative ring. Let $M$ be a finitely generated $R$-module. Then $\text{Supp}(M)$ is closed in $\text{Supp}(R)$.

Proof. Suppose $M = Rm_1 + \cdots + Rm_k$. Then $M_p \neq 0$ iff there exists $1 \leq j \leq k$ such that $m_j \neq 0$ in $M_p$ (i.e. $\text{Ann}(m_j) \subset p$) iff $\text{Ann}(M) = \bigcap_{i=1}^k \text{Ann}(m_i) \subset p$. Then $\text{Supp}(M)$ is the closure of $\text{Ann}(M)$ in $\text{Spec}(A)$. □

5.11 Lemma
If $B$ is unramified over a commutative ring $A$, then $\text{Spec}(\mu): \text{Spec}(B) \rightarrow \text{Spec}(B \otimes_A B)$ is an open immersion\(^7\). (The converse of this is also true, but we do not need it at the moment.)

Proof. Let $\Delta = \text{Spec}(\mu)$ is the diagonal map. Take $p \in \text{im}(\Delta)$, by Lemma 5.9, $I/I^2 = 0$. Then $I_p/I_p^2 = (I/I^2)_p = 0$. Since $p \supset \mu^{-1}(0) = I$, $I_p^2 \subset p(B \otimes_A B)_p I_p$. Then $I_p$ is a finitely generated $B \otimes_A B_p$-module. By Nakayama lemma, $I_p = 0$.

By Lemma 5.10, $\text{Supp}(I)$ is closed in $\text{Spec}(B \otimes_A B)$, there exists $f \in B \otimes_A B \setminus p$ such that $I_f = 0$. For all $q \not\in f$, $I_q = (I_f)_q = 0$. Hence $q \supset I$, $q \in \text{im}(\Delta)$. □

5.12 Lemma
Let $A$ be a local ring. Let $B$ be an unramified $A$-algebra. Let $C$ be an $A$-algebra with a prime ideal $p$. Let $i: C \rightarrow k(p) = \frac{C_p}{pC_p}$ be the canonical morphism. Suppose $u, v: B \rightarrow C$ are $A$-morphisms such that $i \circ u = i \circ v$. Then there exists $f \in C \setminus p$ with a canonical morphism $j: C \rightarrow C_f$ such that $j \circ u = j \circ v$.

\(^7\)Open immersion: injective and the image is open in the target.
Proof. Morphisms \( \text{Spec}(u), \text{Spec}(v) : \text{Spec}(C) \to \text{Spec}(B) \) over \( \text{Spec}(A) \) provides \( \text{Spec}(C) \to \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(B) \cong \text{Spec}(B \otimes_A B) \). The Cartesian diagram

\[
\begin{array}{ccc}
W & \rightarrow & \text{Spec}(B) \\
\downarrow \text{Spec}(u) & & \downarrow \Delta \\
\text{Spec}(C) & \rightarrow & \text{Spec}(B \otimes_A B)
\end{array}
\]

means that \( W = \{ \mathfrak{p} \in \text{Spec}(C) \mid u^{-1}(\mathfrak{p}) = v^{-1}(\mathfrak{p}) \in \text{Spec}(B) \} \) and \( \iota \) is inclusion. By Lemma 5.11, \( \Delta \) is an open immersion, then \( \iota \) is an open immersion. Since \( i \circ u = i \circ v \), \( u^{-1}(\mathfrak{p}) = \text{ker}(i \circ u) = \text{ker}(i \circ v) = v^{-1}(\mathfrak{p}) \), i.e. \( \mathfrak{p} \in \iota(W) \). Then there exists \( f \in C \setminus \mathfrak{p} \) such that \( D(f) \subset \iota(W) \). Therefore

\[
\left( \text{Spec}(u), \text{Spec}(v) \right) \circ \text{Spec}(j) = \left( \left( \text{Spec}(u), \text{Spec}(v) \right) \circ \iota \right) \circ D(f) = \left( \Delta \circ \text{Spec}(u) \right) \circ D(f)
\]

We obtain \( \text{Spec}(u) \circ \text{Spec}(j) = \text{Spec}(v) \circ \text{Spec}(j) \). Hence \( j \circ u = j \circ v \).

\[ \square \]

5.13 Lemma

Let \( (A, m, k) \) is a local ring. Let \( B, C \) be essentially étale \( A \)-algebras with residue fields \( k_B, k_C \) respectively. Then \( \text{Hom}_A(B, C) \to \text{Hom}_k(k_B, k_C) \) is injective.

Proof. Let \( \pi_B : B \to k_B \) and \( \pi_C : C \to k_C \) be canonical quotient maps. Suppose \( u, v \in \text{Hom}_A(B, C) \) such that \( \overline{u} = \overline{v} \). From the commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{u,v} & C \\
\downarrow \pi_B & & \downarrow \pi_C \\
k_B & \xrightarrow{k_B \cdot v} & k_C
\end{array}
\]

we have \( \pi_C \circ u = \pi \circ \pi_B = \overline{u} \circ \pi_B = \pi_C \circ v \). By Lemma 5.12, there exists \( f \in C \setminus \mathfrak{p} \) with a canonical \( j : C \to C_f \) such that \( j \circ u = j \circ v \). Since \( C \) is local, \( f \in C^* \). We have \( C_f = C \), \( j = \text{Id}_C \) and \( u = v \).

\[ \square \]

5.14 Lemma

Let \( (A, m, k) \) is a local ring. Let \( B \) be an essentially étale \( A \)-algebras with residue field \( k_B \). Let \( H \) be a Henselian local ring with residue field \( k_H \). Then \( \text{Hom}_A(B, H) \to \text{Hom}_k(k_B, k_H) \) is surjective.

Proof. We have \( \text{Hom}_A(B, H) \cong \text{Hom}_H(B \otimes_A H, H) \). Similarly, \( \text{Hom}_A(B, k_H) \cong \text{Hom}_k(k_B \otimes_A k, k_H) \). \( \text{Hom}_A(B, k_H) \cong \text{Hom}_H(B \otimes_A H, k_H) \). It suffices to show that \( \text{Hom}_H(B \otimes_A H, H) \to \text{Hom}_H(B \otimes_A H, k_H) \) is surjective. For \( f : B \otimes_A H \to k_H, n = \text{ker}(f) \) is a maximal ideal of \( B \otimes_A H \) over \( m_H \) such that \( (B \otimes_A H)/n \cong k_H \). By Theorem 3.5(5), \( (B \otimes_A H)_n \cong H \). Then the canonical map \( B \otimes_A H \to (B \otimes_A H)_n \cong H \) is a lift of \( f \).

\[ \square \]

5.15 Theorem

Let \( (A, m, k) \) be a henselian local ring. Then \( B \to \mathfrak{B} \) from the category of finite étale local \( A \)-algebras to the category of finite separable extensions of \( k \) is an equivalence of categories.

Proof. Fully-faithful. Let \( B, C \) be finite étale local \( A \)-algebras. Then they are essentially étale. By Review 3.2(1), \( C \) is Henselian. Then it follows from Lemma 5.13 and Lemma 5.14 that \( \text{Hom}_A(B, C) \to \text{Hom}_k(k_B, k_C) \) is bijective.
Essentially surjective. Let $k'$ be a finite separable extension of $k$. By primitive element theorem, there exists a separable polynomial $\overline{f} \in k[T]$ such that $k' \simeq \frac{k[T]}{(f(T))}$. Lift $\overline{f}$ to $f \in A[T]$. Then $\frac{A[T]}{(f(T))}$ is standard étale over $A$ with residue field $k'$.

5.16 Corollary

Let $(A, m, k)$ be a henselian local ring. Then $B \mapsto \overline{B}$ from the category of finite étale $A$-algebras to the category of finite products of finite separable extensions of $k$ is an equivalence of categories.

Proof. Finite product commutes with everything in the previous theorem. □
6. June 15 Properties of Henselization

6.1 Theorem
A^h is henselian.

Proof. We verify Theorem 3.5(5). Let B be an étale algebra over A^h with a maximal ideal n lying over mA^h such that B/n ∼= k, we need to show the canonical A^h ∼= B_n. Since B is étale over A^h, there exists a ∊ A^h \ mA^h and b ∊ B \ n such that B_b is standard étale over (A^h)_a = A^h. Suppose B_b ∼= \frac{A^h[T]}{(f(T))^g} for some f, g ∊ A^h[T] such that f' is invertible in B_b. Suppose A^h = \lim_{\lambda} A_\lambda where A_\lambda are strictly essentially étale A-algebras with \varphi: A_\lambda → A^h, there exists a sufficiently large \lambda and f_\lambda, g_\lambda ∊ A_\lambda[T] such that \varphi_\lambda(f_\lambda) = f and \varphi_\lambda(g_\lambda) = g. Let B_\lambda = \frac{A_\lambda[T]}{(f_\lambda(T))^ \lambda}. Let \psi_\lambda: B_\lambda → B_b be the extension of \varphi_\lambda and p = \psi_\lambda^{-1}(n). Since p lies over mA_\lambda, k = \frac{A_\lambda}{mA_\lambda} → B_\lambda/p. Also, \psi_\lambda induces B_\lambda/p → B/n ∼= k. Then B_\lambda/p ∼= k. Hence (B_\lambda)_p is strictly essentially étale over A_\lambda. By Lemma 4.3, (B_\lambda)_p is strictly essentially étale over A. Then there exists m ∊ A such that (B_\lambda)p ∼= A_\mu. We have a Cartesian diagram of canonical morphisms

\[ \begin{array}{ccc} \Spec(B_\lambda) & \xrightarrow{\Delta} & \Spec(B_\lambda \otimes_{A_\lambda} B_\lambda) \\ \downarrow & & \downarrow \\ \Spec(A_\mu) & \xrightarrow{\Delta} & \Spec(B_\lambda \otimes_{A_\lambda} A_\mu) \end{array} \]

Since B_\lambda is étale over A_\lambda, B_\lambda is unramified over A_\lambda. By Lemma 5.11, Δ is an open immersion. Since Spec(B_\lambda) is separated, Δ is also a closed immersion⁸. Let B_\mu = B_\lambda \otimes_{A_\lambda} A_\mu, \psi_\mu: B_\mu → B_\lambda and q = \psi^{-1}_\mu(n). By Lemma 3.3, A_\mu → B_\mu provides an isomorphism A_\mu ∼= (B_\mu)_q. Therefore

\[ A^h ∼= A_\mu \otimes_{A_\mu} A^h ∼= (B_\mu)_q \otimes_{A_\mu} A^h = (B_\lambda \otimes_{A_\lambda} A_\mu)_q \otimes_{A_\mu} A^h ∼= (B_\lambda \otimes_{A_\lambda} A^h)_n ∼= (B_b)_n ∼= B_n. \]

□

6.2 Theorem
Let (A, m, k) be a local ring. Let (A^h, mA^h, k) be its henselization with the canonical local homomorphism i: A → A^h. For all henselian local ring H with a local homomorphism j: A → H, there exists a unique local homomorphism j': A^h → H such that j' ∘ i = j, i.e. a commutative diagram

\[ A \xrightarrow{i} A^h \xrightarrow{j} H \xrightarrow{j'} \]

Proof. By Lemma 5.4, Lemma 5.13 and Lemma 5.14, we have for all λ

\[ \Hom_{\text{loc}, A}(A_\lambda, H) ∼= \Hom_A(A_\lambda, H) ∼= \Hom_k(k, k_H). \]

⁸A morphism \( f: X \to Y \) is a closed immersion if \( f: X \to f(X) \) is a homeomorphism; \( f(X) \) is closed in \( Y \); and \( \mathcal{O}_Y \to f_*\mathcal{O}_X \) is surjective.
Since \( \text{Hom}_k(k, k_H) \) is a singleton, \( \text{Hom}_A(A_\lambda, H) \) is a singleton for all \( \lambda \). By Lemma 5.4 and the universal property of inductive limit, \( \text{Hom}_{\text{loc}, A}(A^h, H) \simeq \text{Hom}_A(A^h, H) \) is a singleton.

### 6.3 Lemma

Let \( A_\lambda \) be an inductive system of local rings with limit \( A' = \lim A_\lambda \). Suppose \( f_{\lambda \mu}: A_\lambda \to A_\mu \) is flat for all \( \lambda < \mu \). Then \( f_\lambda: A_\lambda \to A' \) is flat.

**Proof.** Fix \( f_\lambda: A_\lambda \to A' \) is flat. Let \( A = \{a\} \) be a singleton. □

### 6.4 Theorem

\( A^h \) is a faithfully flat \( A \)-module.

**Proof.** By Lemma 6.3, \( A^h \) is flat over \( A \). For all \( A \)-module \( M \neq 0 \), we take \( 0 \neq x \in M \). Since \( mA^h \neq A^h \), by Nakayama lemma, \( M \otimes_A \frac{A^h}{mA^h} \neq 0 \). Hence

\[
M \otimes_A A^h \supset Ax \otimes A A^h \simeq \frac{A}{\text{Ann}(x)} \otimes_A A^h \supset \frac{A}{m} \otimes_A A^h \simeq k \neq 0.
\]

Therefore \( M \otimes_A A^h \neq 0 \), \( A^h \) is faithfully flat over \( A \). □

### 6.5 Lemma

Let \( R \) be a noetherian commutative ring with an ideal \( I \). Let \( M \) be an \( R \)-module. Suppose the \( I \)-adic topology of \( R \) is Hausdorff. Then \( M \) is flat over \( R \) iff \( M/I^nM \) is flat over \( R/I^n \) for all \( n \geq 1 \).

**Proof.** See [Bou89, Ch. III, § 5, no. 2, Th. 1]. □

### 6.6 Theorem

\( A^h \) is noetherian iff \( A \) is noetherian.

**Proof.** Suppose \( A^h \) is noetherian. Let \( a_1 \subseteq a_2 \subseteq \cdots \) be a sequence of ideals of \( A \). Then \( a_1 \otimes A A^h \subseteq a_2 \otimes A A^h \subseteq \cdots \) is stationary, i.e. there exists \( N > 0 \), for all \( n > N \), \( a_n \otimes A A^h = a_{n+1} \otimes A A^h \). By Theorem 6.4, \( A^h \) is faithfully flat over \( A \). Then \( a_n = a_{n+1} \) for all \( n > N \). Therefore \( A \) is noetherian.

Conversely, suppose \( A \) is noetherian. Since \( A_\lambda \) is (strictly) essentially étale over \( A \), \( A_\lambda \simeq B_n \) for some étale \( A \)-algebra \( B \) with maximal ideal \( n \) over \( m \). Since \( B \) is of finite presentation over \( A \), \( B \) is noetherian. Since any localization of a noetherian ring is noetherian, \( A_\lambda \) is noetherian for all \( \lambda \).
Next, the m-adic topology on $A_\lambda$ is Hausdorff. Suppose $I = \bigcap_{n=1}^{\infty} (mA_\lambda)^n$. Then $(mA_\lambda)I = I$. Since the local ring $(A_\lambda, mA_\lambda, k)$ is a noetherian, $I$ is finitely generated. By Nakayama lemma, $I = 0$.

Now, the m-adic topology on $A^h$ is Hausdorff. Suppose $x \in \bigcap_{n=1}^{\infty} (mA^h)^n$ and $x = f_\lambda(x)$ for some $x_\lambda \in A_\lambda$. By Lemma 5.1, $A^h_\lambda = A^h$. By Theorem 6.4, $A^h$ is faithfully flat over $A_\lambda$. Then $f_\lambda^{-1}((mA^h)^n) = (mA_\lambda)^n$ for all $n \geq 19$. We have

$$x_\lambda \in f_\lambda^{-1}\left(\bigcap_{n=1}^{\infty} (mA^h)^n\right) = \bigcap_{n=1}^{\infty} f_\lambda^{-1}((mA^h)^n) = \bigcap_{n=1}^{\infty} (mA_\lambda)^n.$$

By the previous paragraph, $x_\lambda = 0$. Hence $x = 0$, $\bigcap_{n=1}^{\infty} (mA^h)^n = 0$.

Let $\widehat{A^h} = \lim_{\longleftarrow} \frac{A^h}{(mA^h)^n}$ be the completion of $A^h$ with respect to m-adic topology. Then $(\widehat{A^h}, mA^h, k)$ is local.

Next, we show that $\widehat{A^h}$ is noetherian. Since $A^h$ is flat over $A$,

$$\frac{(mA^h)^n}{(mA^h)^{n+1}} \simeq \frac{m^n}{m^{n+1}} \otimes_A A^h \simeq \frac{m^n}{m^{n+1}} \otimes_k (k \otimes_A A^h) \simeq \frac{m^n}{m^{n+1}} \otimes_k k \simeq \frac{m^n}{m^{n+1}}$$

Then $\text{gr}(\widehat{A^h}) \simeq \text{gr}(A^h) \simeq \bigoplus_{n=0}^{\infty} \frac{m^n}{m^{n+1}}$ is a $k$-algebra of finite type since $A$ is noetherian. By Hilbert basis theorem and that homomorphic images preserves noetherian property, $\text{gr}(\widehat{A^h})$ is noetherian.

Let $I$ be an ideal of $\widehat{A^h}$. Then $I \cap (\widehat{A^h})^n \to (\widehat{A^h})^n$ is an injection and $\text{gr}(I)$ is a finitely generated ideal of $\text{gr}(A^h)$. Suppose its generators are $x_1, \ldots, x_r$ where $x_i \in I \cap (\widehat{A^h})^n$. Let $F^i = \widehat{A^h}$ with m-adic filtration $F^i = F_0^i \supset F_1^i \supset F_2^i \supset \cdots$ where $F_n^i = (mA^h)^{n+n}$ for all $n \geq 1$. Let $F = \bigoplus_{i=1}^{r} F^i$. Then $F$ is a free $\widehat{A^h}$-module of rank $r$ with filtration $F = F_0 \supset F_1 \supset F_2 \supset \cdots$ where $F_n = \bigoplus_{i=1}^{r} F_n^i$. Let $\phi: F \to I$ be the map defined by $\phi(e_i) = x_i$ where $(e_1, \ldots, e_r)$ is the standard basis of $F$. Then we have

$$0 \xrightarrow{\phi} \frac{F}{F_n} \xrightarrow{\phi} \frac{F}{F_{n+1}} \xrightarrow{\phi} \cdots \xrightarrow{\phi} \frac{F}{F_n} \xrightarrow{\phi} 0$$

$$0 \xrightarrow{0} \frac{I \cap (\widehat{A^h})^n}{I \cap (\widehat{A^h})^{n+1}} \xrightarrow{\phi} \cdots \xrightarrow{\phi} \frac{I \cap (\widehat{A^h})^n}{I \cap (\widehat{A^h})^{n+1}} \xrightarrow{\phi} 0$$

\footnote{An $\phi: A \to B$ is faithfully flat iff $\phi^{-1}(IB) = I$ for all ideal $I$ of $A$.}

\footnote{If a commutative ring $R$ is noetherian, then $A[T]$ is noetherian.}
here $\text{gr}(\phi)_n(x_i \mod F_{n+1}) = \underline{x}_i$ for all $x_i \in F_n$. Then $\text{gr}(\phi)_n$ is surjective for all $n \geq 0$. Clearly, $\phi_k : 0 \to 0$ is surjective. If $\phi_n$ is surjective, by snake’s lemma, $\phi_{n+1}$ is surjective. Hence surjections $(\phi_n)_{n \geq 0}$ define a surjection $\hat{\phi} : \hat{F} \to \hat{I}$.

Since $\hat{A}^h$ is complete, the $m$-adic topology of $\hat{A}^h$ is Hausdorff. Then canonical map $I \to \hat{I} = \varprojlim I \cap (mA^h)^n$ is an injection. By the functoriality,

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & I \\
\downarrow & & \downarrow \\
\hat{F} & \xrightarrow{\hat{\phi}} & \hat{I}
\end{array}
\]

Then $\hat{\phi}$ is surjective and hence $I$ is finitely generated.

Then, we show that $\hat{A}^h$ is flat over $A^\lambda$ for all $\lambda$.

\[
\begin{array}{cccccccc}
0 & \to & (mA^h)^n & \to & A^h & \to & A^h & \to & 0 \\
0 & \to & (mA^h)^n & \to & \hat{A}^h & \to & \hat{A}^h & \to & 0
\end{array}
\]

Since $\frac{A^h}{mA^h} = k = \frac{\hat{A}^h}{mA^h}$, by snake’s lemma and induction, $\frac{A^h}{(mA^h)^n} \simeq \frac{\hat{A}^h}{(m\hat{A}^h)^n}$ for all $n \geq 1$. Since $\frac{A^h}{(mA^h)^n} \simeq \frac{A^h}{(mA^h)^n}$ and $A^h$ is flat over $A^\lambda$, $\frac{\hat{A}^h}{(m\hat{A}^h)^n}$ is flat over $\frac{A^h}{(mA^h)^n}$ for all $n \geq 1$. By Lemma 6.5, $\hat{A}^h$ is flat over $A^\lambda$.

Finally, we show that $\hat{A}^h$ is noetherian. By Lemma 6.3, $\hat{A}^h$ is flat over $A^h$. Since $m\hat{A}^h \subseteq \hat{A}^h$, $\hat{A}^h$ is faithfully flat over $A^h$. Since $\hat{A}^h$ is noetherian, similar to the first paragraph, $\hat{A}^h$ is noetherian. \qed

References


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